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On the geometrical form of volcanoes

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Many volcanic edifices have a remarkably symmetric geometrical form. An example is Mount Fuji in Japan. We model this form assuming that the surface of the volcano is a surface of uniform hydraulic potential; that an erupting magma will follow the path of minimum resistance to the surface. In order to model the resistance to fluid flow we assume the volcanic edifice is a uniform porous medium. The vertical flow of magma is also resisted by the gravitational body force. If the volcano becomes too tall flank eruptions will widen it; if the volcano becomes too wide summit eruptions will increase its elevation. Using the Dupuit approximation for an unconfined aquifer it is shown that the percolation equation is applicable. As magma reaches the surface it is assumed to extend the solid, porous matrix. A similarity solution is obtained to this moving boundary problem. The solution predicts a uniform shape for all volcanoes. This shape is shown to be in excellent agreement with the geometrical form of Mount Fuji.

1. Introduction

The geometrical form of many volcanic edifices exhibits a remarkable symmetry. A large fraction of the composite volcanoes associated with subduction zones have a near constant slope on their flanks and a form that is concave upwards near their summits. Examples include Mount Fuji in Japan and Mount Mayan in the Philippines. The large shield volcanoes of the Hawaiian Islands also exhibit circular symmetry with a near constant slope. Moana Loa is probably the best example. It should be emphasized, however, that a number of phenomena can lead to non-symmetrical edifices. These include parasitic centers of volcanism on the flanks, glacial and other types of erosion, and explosive eruptions.

John Milne travelled to Japan in 1875 to become Professor of Mining Engineering at the Imperial Institute of Technology. He was fascinated by the geometrical form of the composite volcanoes of central Japan. He proposed [1,2] that the observed form was the result of slope stability. He applied soil mechanics theory and carried out experiments with granular material to reproduce the observed form. His approach is likely to be applicable to cinder cones or small volcanoes constructed primarily of ash flows but not to edifices composed of a large number of lava flows. Becker [3] extended the analysis to slope stability based on rock mechanics.

In this paper we propose that the hydraulic resistance to the flow of magma determines the geometrical form of volcanoes. The surface of the volcano is a surface of constant hydraulic potential. In order to model the flow of magma through the volcanic edifice we assume it is a uniform porous material. Each flow passes through the interior of the edifice, reaches the surface, and extends the surface as it solidifies.

2. Model

We assume that a volcanic edifice is the constructional sum due to the solidification of many small lava flows. The way in which these flows extend the edifice is illustrated qualitatively in Fig. 1. In this idealized model magma reaches the center of the base of the edifice through a volcanic pipe. It is recognized from studies of groundwater migration that volcanic edifices are permeated by fractures and joints. Presumably these are primarily thermal contraction cracks formed during the cooling of individual flows. In some volcanoes the permeability of the structure is dominated by rift zones and most surface volcanic flows emanate from these rift zones.

We assume that the magma is driven through the pre-existing matrix of channels in search of the least resistant path to the surface. This is illustrated in Fig. 1a. In Fig. 1b the magma at one point reaches the surface. *This will be the point on the surface where the hydraulic resistance to the flow is a minimum*. The magma reaching the surface will result in a surface flow which covers part of the surface extending the volcanic edifice (Fig. 1c). The flow will increase the hydraulic resistance of that part of the edifice and the next eruption will occur at another point on the surface. If a volcano grows too tall flank eruptions will widen it; if a volcano grows too wide summit eruptions will increase its elevation. The equal resistance to flow requires that the volcano grows symmetrically as many individual eruptions are averaged.

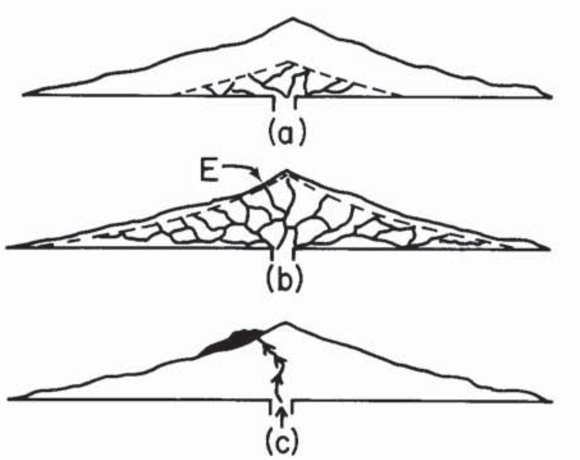

Fig. 1. Migration of magma through a volcanic edifice during an eruption. (a) Magma migrates outward from the source pipe at the center of the volcano, the dashed line represents the equipotential surface of the advancing magma. (b) The magma reaches the surface at the point of minimum resistance to the flow. (c) The volcanic edifice is extended by the surface flow.

To model the hydraulic resistance of the volcanic edifice we assume that it has a uniform hydraulic permeability. The flexure of the initial surface due to the load of the volcano is neglected. In order to simplify the analysis we assume that the slope of the volcanic edifice is small, $\partial h/\partial r << 1$, where h is the height of the volcano as a function of the radial coordinate r. The problem is essentially the same as the flow in a porous, unconfined aquifer. The small slope assumption allows the use of the Dupuit approximation [4] and the radial flow of magma Q_r is given by:

$$Q_r = 2\pi r h u_r = -\frac{2\pi k \rho_m g r h}{\mu}\frac{\partial h}{\partial r} \tag{1}$$

where u_r is the radial Darcy velocity (the mean velocity per unit area, not the flow velocity in magma tubes and channels), k is the permeability, ρ the magma density, μ the magma viscosity, and g the acceleration of gravity.

In accordance with our hypothesis Q_r is the mean radial flow of magma averaged over a large number of eruptions. Since the inertia of the magma is negligible it is appropriate to apply a steady-state analysis to this problem even though each actual flow only penetrates a small fraction of the porous edifice.

The permeability k is the actual permeability of the magma tubes and channels during the eruption. A radial mass balance requires that:

$$2\pi r\frac{\partial h}{\partial t} = -\frac{\partial Q_r}{\partial r} \tag{2}$$

The volcano grows in elevation at the expense of the radial flow. Substitution of (1) into (2) gives:

$$\frac{\partial h}{\partial t} = \frac{k\rho_m g}{\mu r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right) \tag{3}$$

This is known as the perculation or Boussinesq equation. We assume that as the magma reaches the surface it solidifies and becomes part of the porous matrix.

3. Methods of solution

In order to solve the perculation equation for this problem we introduce the following similarity variables:

$$f=\left(\frac{k\rho_m g}{\mu Q_0}\right)^{1/2} h \tag{4}$$

$$\xi=\left(\frac{\mu}{k\rho_m g Q_0}\right)^{1/4} \frac{r}{t^{1/2}} \tag{5}$$

where Q_0 is the value of Q_r at $r=0$, it is the magma supply rate. Substitution of (4) and (5) into (3) gives:

$$ff''+(f')^2+\frac{1}{\xi}ff'+\tfrac{1}{2}\xi f'=0 \tag{6}$$

where $f'=\mathrm{d}f/\mathrm{d}\xi$.

The boundary condition on the magma supply requires that:

$$-\frac{2\pi k\rho_m g r h}{\mu}\frac{\partial h}{\partial r}\to Q_0 \text{ as } r\to 0 \tag{7}$$

In terms of the similarity variables, (4) and (5), this can be written:

$$\xi ff'\to -\frac{1}{2\pi} \text{ as } \xi\to 0 \tag{8}$$

The condition that the flowing magma adds to the structure of the edifice makes this a moving boundary problem. The boundary condition is the same as that applied to the Stefan problem [5]. We denote the radius of the base of the volcanic edifice as r_0. From (5) we obtain:

$$r_0=\left(\frac{k\rho_m g Q_0}{\mu}\right)^{1/4}\xi_0 t^{1/2} \tag{9}$$

where ξ_0 is the non-dimensional radius of the volcano. From (9) we see that the width of the volcano grows proportional to the square root of time. Taking the time derivative of (9) we obtain:

$$u_0=\frac{\partial r_0}{\partial t}=\frac{1}{2}\left(\frac{k\rho_m g Q_0}{\mu}\right)^{1/4}\frac{\xi_0}{t^{1/2}} \tag{10}$$

However, the velocity of the edge of the volcano is also given by (1):

$$u_0=-\frac{k\rho_m g}{\mu}\frac{\partial h}{\partial r} \tag{11}$$

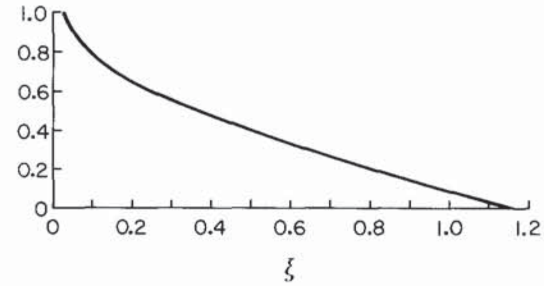

Fig. 2. Dependence of the non-dimensional height on the non-dimensional radius.

And introduction of the non-dimensional variables, (4) and (5) gives:

$$u_0=-\left(\frac{k\rho_m g Q_0}{\mu}\right)^{1/4}\frac{f'}{t^{1/2}} \tag{12}$$

Equating (10) and (12) gives:

$$f'=-\tfrac{1}{2}\xi \text{ at } \xi=\xi_0 \tag{13}$$

but $H=0$ at $r=r_0$ so that:

$$f=0 \text{ at } \xi=\xi_0 \tag{14}$$

The required boundary conditions on (6) are given by (7), (13), and (14).

Since (6) is a rather complicated ordinary differential equation a numerical solution is required. The solution is obtained by guessing a value for ξ_0; f and f' at $\xi=\xi_0$ are given by (13) and (14), f'' is determined from (6), new values of f' and f at $\xi=\xi_0-\delta\xi$, f'' $(\xi_0-\delta\xi)$ is determined from (6), and the process is repeated to determine f, f', and f'' as a function of ξ. The process is repeated for various values of ξ_0 until (8) is satisfied. We find that $\xi_0=1.16$. The solution for f as a function of ξ is given in Fig. 2. Near the origin the solution has a weak logarithmic singularity. Since the Dupuit approximation requires that the slope be small, the solution is not valid in the vicinity of the origin. In this region a two-dimensional solution is required.

4. Results

The profile given in Fig. 1, although universal, can be scaled vertically an arbitrary amount. The

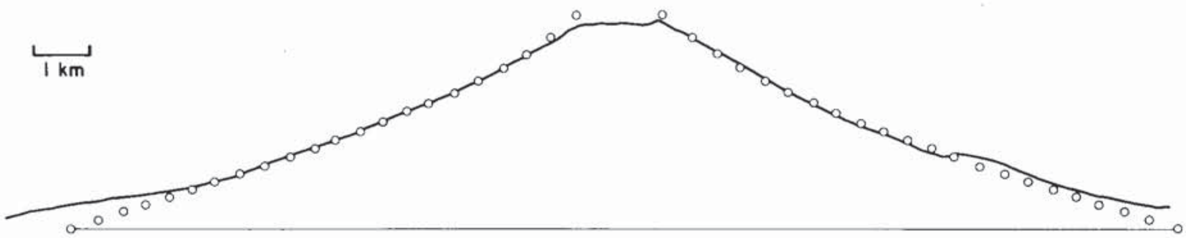

Fig. 3. Comparison of the porous flow theory (circles) with the geometrical form of Mount Fuji, Japan (solid line).

flank slope can be adjusted until a best fit is obtained. Mount Fuji in Japan is considered to have the typical form of a stratovolcano. In Fig. 3 a cross section of Mount Fuji is compared with our universal profile. In general the agreement is quite good. Near the base the observed profile is more rounded. This can be attributed to deposits of alluvium. The theory is not expected to be valid near the summit where the solution is singular.

The above theory also predicts the dependence of such quantities as volcanic radius and flank slope on the parameters. Taking $\xi_0 = 1.16$ the radius of the volcano from (9) is:

$$r_0 = 1.16\left(\frac{k\rho_m g Q_0}{\mu}\right)^{1/4} t^{1/2} \tag{15}$$

From (4) and (5) the slope is given by:

$$\frac{\partial h}{\partial r} = \left(\frac{\mu}{k\rho g}\right)^{3/4} Q_0^{1/4} \frac{f'}{t^{1/2}} \tag{16}$$

And from (13) the slope at $\xi = \xi_0$ is:

$$\left(\frac{\partial h}{\partial r}\right)_0 = -0.58\left(\frac{\mu}{k\rho g}\right)^{3/4} \frac{Q_0^{1/4}}{t^{1/2}} \tag{17}$$

Taking the negative product of (15) and (17) gives a reference height for the volcano:

$$h_r = 0.673\left(\frac{\mu Q_0}{k\rho g}\right)^{1/2} \tag{18}$$

We see from our analysis that the reference height is independent of time. Our theory suggests that volcanoes grow primarily by an increase in their radius once they reach the critical height. And the radius increases with the square root of time.

Since the reference height from (18) is a function of flow rate and permeability, it is not possible to draw absolute conclusions regarding the dependence of height on such parameters as gravity and viscosity. The flow rate will depend on the hydraulic head available to drive magma through the edifice. It has been suggested [6] that this head depends upon the thickness of the lithosphere beneath the volume. If, however, the flow rate and permeability are constant we conclude from equation (18) that the reference height of volcanoes depends upon $g^{-1/2}$. The ratio of the gravity field on Mars to that on earth is 0.38. Therefore, it is predicted that the ratio of volcanic height on Mars to the height on the earth is 1.62. The height of the highest volcanoes on Mars is about 21 km above the Mars reference surface. The height of the Hawaiian shield volcanoes is about 10 km with respect to the sea flow. Although many factors undoubtedly influence the height of volcanoes, the influence of gravity and viscosity on the flow through the volcanic edifice may play an important role.

We believe that the simple, symmetrical form of many volcanic edifices indicates that a simple physical mechanism may dominate the construction of such edifices. In this paper we propose a porous flow model to represent the resistance to the flow of magma through the edifice. The surface of the volcano is predicted to be an equipotential surface for the flow of magma through a uniform porous medium. A similarity solution to the problem is obtained which predicts a uniform shape for volcanoes.

Clearly much work remains to be done in refining the proposed model. A two-dimensional numerical solution for the flow would remove the singular behavior near $r = 0$. For large volcanic edifices the weight of the overburden may reduce the permeability at depth. Such a decrease with depth can be incorporated into the model.

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